CONTINUOUS REDUCIBILITY AND DIMENSION OF METRIC SPACES

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Abstract. If (X, d) is a Polish metric space of dimension 0, then by Wadge's Lemma, there are at most two Borel subsets that are incomparable with respect to continuous reducibility. In contrast, for any metric space (X, d) of positive dimension, we prove that there is a perfect set of Borel codes for Borel subsets that are pairwise incomparable with respect to continuous reducibility. This implies equivalence of the following conditions for Polish metric spaces (X, d): (a) (X, d) has dimension 0 (b) The Wadge order on the Borel subsets of X satisfies the semi-linear ordering principle (c) The Wadge order on the Borel subsets of X is a well-quasiorder.

Contents

1. Introduction

The *Wadge order* on subsets of a topological space refines several important hierarchies, for instance the Borel hierarchy and the difference hierarchy. If A and B are subsets of a topological space (X, τ) and there is a continuous map $f : X \to X$ with $A = f^{-1}[B]$, then A is *continuously reducible* to B with respect to (X, τ), A is below B in the Wadge order and $A \leq_W B$. This classifies subsets of Polish spaces of dimension 0 according to their complexity. The fine structure of the Wadge order on the Cantor space and the Baire space has been intensely studied [Wad12, Lou83, And06]. Some of these results have been extended to other classes of functions [AM03, MR09, MR10b, MR10a]. For more restricted classes of functions, some of these results fail, for instance for Lipschitz reducibilities on ultrametric Polish spaces [MR14, MRS14].

This paper is part of a program to understand the structure of Borel subsets of arbitrary Polish spaces as ordered by the Wadge order (see [MRSS15]). The Wadge order on the Borel subsets of the Cantor space and the Baire space satisfies at least two important conditions. The first is well-foundedness (see [And07, Theorem 8]) and the second is the semi-linear ordering principle (SLO, see [And07]). These conditions are also satisfied for all other Polish spaces of dimension 0. However, they fail for some Polish spaces of positive dimension. For instance, the Wadge order can be ill-founded [Her96]. The program to study the Wadge order of arbitrary Polish spaces was initiated in [MRSS15]. This continues work in [Her96] and [Sel05a] and is motivated by [And07, MR14, MRS14] and questions and remarks in [Ste80, Woo10].

The semi-linear ordering principle for Borel subsets of the Baire space follows from Wadge's Lemma.

Lemma 1.1 (Wadge). *Suppose that A, B are Borel subsets of the Baire space ${}^{\omega}\omega$. Then there is a continuous function $f : {}^{\omega}\omega \to {}^{\omega}\omega$ such that $A = f^{-1}[B]$ or $B = f^{-1}[{}^{\omega}\omega \setminus A]$.*

The semi-linear ordering principle can fail for other Polish spaces, since there may be more than two Borel subsets that are not comparable in the Wadge order by a result of Andretta. Moreover, the Wadge order is well-founded for some Polish metric spaces that do not have dimension 0 [MRSS15, Remark after Theorem 5.15]. In this space, any two subsets that are non-trivial, i.e.

Date: May 2, 2016.

nonempty and not equal to the whole space, are not comparable in the Wadge order, since every continuous function on the space is constant or the identity.

Motivated by work of Andretta [And07], Motto Ros [MR10b], Selivanov [Sel05b] and joint work [MRSS15], the question arises whether the first property fails for all Polish metric spaces of positive (small inductive) dimension. The following result answers this question.

Theorem 1.2. *Suppose that (X, d) is a metric space of positive (small inductive) dimension. Then there is a perfect set of Borel codes for distinct subsets of X that are pairwise incomparable with respect to continuous reducibility.*

This result implies that Wadge's Lemma 1.1 fails for any metric space (X, d) of positive (small inductive) dimension. The sets that are constructed in the proof are intersections of open and closed sets. By a Borel code, we mean a sequence of open balls which codes a Borel set in a canonical way.

Since every separable regular Hausdorff space is metrizable by Urysohn's metrization theorem, Theorem 1.2 holds for these spaces. The result does not hold for all separable spaces [MRSS15, Proposition 5.33].

We will see below that Theorem 1.2 is optimal in the sense that it is not possible to prove the existence of other configurations with respect to the Wadge order of Borel subsets for all Polish metric spaces with positive (small inductive) dimension.

We can use Theorem 1.2 to characterize dimension 0 for Polish spaces as follows. The first property of the Wadge order in the following definition has been studied in [And07]. The second property is an important notion in the classifcation of quasi-orders [CP14].

Definition 1.3. (a) Suppose that A is a collection of subsets of a metric space (X, d). The Wadge order on A satisfies the *semi-linear ordering principle (SLO)* if Wadge's Lemma holds for elements of A.

(b) A quasi-order $\langle A, \leq \rangle$, i.e. a transitive reflexive relation, is a *well-quasiorder* if it satisfies the following conditions.
   (i) There is no infinite strictly $\leq$-descending sequence $\vec{x} = \langle x_n \mid n \in \omega \rangle$ in A.
   (ii) There is no infinite sequence $\vec{x} = \langle x_n \mid n \in \omega \rangle$ in A whose elements x_n are pairwise $\leq$-incomparable.

Theorem 1.2 implies the following characterization.

Theorem 1.4. *Suppose that (X, d) is a Polish metric space. Then the following conditions are equivalent.*

(1) *X has (small inductive) dimension 0.*
(2) *There is no set consisting of more than 2 Borel subsets of X that are pairwise incomparable in the Wadge order.*
(3) *There is no perfect set of Borel codes for Borel subsets of X that are pairwise incomparable in the Wadge order.*
(4) *The Wadge order on the Borel subsets of X satisfies the semi-linear ordering principle (SLO).*
(5) *The Wadge order on the Borel subsets of X is a well-quasi-order (wqo).*

An alternative definition of a Wadge order for Polish spaces of positive (small inductive) dimension 0 is given in [Peq15]. In contrast, this notion defines a well-quasiorder.

We further consider a problem that arises from Theorem 1.2. The number of Borel sets in Theorem 1.2 is maximal for separable metric spaces. If we additionally assume that the space is locally compact, the result can be extended as follows.

Theorem 1.5. *Suppose that (X, d) is a locally compact metric space of positive (small inductive) dimension. Then there is a (definable) injective map that takes sets of reals to subsets of X such that the subsets are pairwise incomparable with respect to continuous reducibility.*

We will use the following notation.

Definition 1.6. Suppose that (X, d) is a metric space and A, B are subsets of X.

(a) A is *(continuously) reducible to B* ($A \leq B$) if there is a continuous map $f : X \to X$ such that $A = f^{-1}[B]$.

(b) A and B are *(continuously) equivalent* if $A \leq B$ and $B \leq A$.
(c) A and B are *(continuously) comparable* if $A \leq B$ or $B \leq A$.
(d) A and B are *(continuously) incomparable* if $A \not\leq B$ and $B \not\leq A$.

Definition 1.7. A topological space (X, τ) has *(small inductive) dimension* 0 if for every x in X and every open set U subset of X containing x, there is an closed-open subset of U containing x.

This notion of dimension 0 coincides with the notion of large inductive dimension 0 and Lebesgue covering dimension 0 for separable metric spaces [Eng89]. Therefore we will write dimension 0 instead of small inductive dimension 0. The condition is strictly stronger that the condition that the space is totally disconnected, i.e. that all connected components contain only one element. This is witnessed by the complete Erdös space [DvM09].

In Section 2, we give proofs of Theorem 1.2 and Theorem 1.4. In Section 3, we give a proof of Theorem 1.5.

2. Incomparable Borel sets

In this section, we first prove Theorem 1.2. Suppose that (X, d) is a metric space of positive dimension. If $x^* \in X$, we denote the open ball with radius r around x^* as $B_r(x^*) = \{x \in X \mid d(x^*, r) < r\}$.

Since (X, d) has positive dimension, there is some $x^* \in X$ such that there is no neighborhood base at x^* that consists of closed-open sets. Then there is some $r > 0$ with the property that there is no closed-open neighborhood of x^* that is contained in $X_0 = B_r(x^*)$. We fix such x^*, r, X_0 and a strictly increasing sequence $\langle r_n \mid n \in \omega \rangle$ of real numbers with supremum r and $r_0 = 0$.

Definition 2.1. Suppose that $n \in \omega$. We define the following sets.[1]

(a) (i) $C_{<n}$ is the set of $x \in X_0$ with $d(x, x^*) < r_n$,
 (ii) $C_{>n}$ is the set of $x \in X_0$ with $d(x, x^*) < r_n$,
 (iii) $C_{\leq n}$ is the set of $x \in X_0$ with $d(x, x^*) \leq r_n$,
 (iv) $C_{\geq n}$ is the set of $x \in X_0$ with $d(x, x^*) \geq r_n$,

(b) (i) $C_{[n,n+1]} = C_{\leq n+1} \cap C_{\geq n}$,
 (ii) $C_{(n,n+1]} = \begin{cases} C_{\leq n+1} \cap C_{>n} & \text{for } n > 0, \\ C_{\leq n+1} & \text{for } n = 0, \end{cases}$
 (iii) $C_{[n,n+1)} = C_{<n+1} \cap C_{\geq n}$,
 (iv) $C_{(n,n+1)} = \begin{cases} C_{<n+1} \cap C_{>n} & \text{for } n > 0, \\ C_{<n+1} & \text{for } n = 0, \end{cases}$

We will use the sets in Definition 2.1 to partition X_0 into blocks. Note that $C_{[0,1)} = C_{(0,1)}$ and $C_{[0,1]} = C_{(0,1]}$.

Lemma 2.2. *Suppose that $\langle I_n \mid n \in \omega \rangle$ is a sequence of intervals such that for each $n \in \omega$, $I_n = [n, n+1]$, $I_n = (n, n+1]$, $I_n = [n, n+1)$ or $I_n = (n, n+1)$. If $\langle I_n \mid n \in \omega \rangle$ is a partition of $[0, \infty)$ or of of $(0, \infty)$, then $\langle C_{I_n} \mid n \in \omega \rangle$ is a partition of X_0.*

Proof. Since $x^* \in C_{(0,1]}$, the claim follows from Definition 2.1. □

Let Even denote the set of even natural numbers and Odd the set of odd natural numbers.

Definition 2.3. Suppose that $\vec{n} = \langle n_i \mid i \in \omega \rangle$ is a strictly increasing sequence of natural numbers with $n_0 = 0$. We define the following sets.

(a) $\text{Even}^{\vec{n}} = \bigcup_{i \in \text{Even}} [n_i, n_{i+1})$,
(b) $\text{Odd}^{\vec{n}} = \bigcup_{i \in \text{Odd}} [n_i, n_{i+1})$.

The following definition will be used to obtain sets that are incomparable with respect to continuous reducibility.

Definition 2.4. Suppose that $\vec{n} = \langle n_i \mid i \in \omega \rangle$ is a strictly increasing sequence of natural numbers with $n_0 = 0$. We define the following sets.

[1] The definition for $n = 0$ avoids cases in the arguments below.

(a) (i) $D^{\vec{n}}_k = \begin{cases} C_{[2k,2k+1)} & \text{if } k \in \text{Even}^{\vec{n}}, \\ C_{(2k,2k+1]} & \text{if } k \in \text{Odd}^{\vec{n}}, \end{cases}$

(ii) $D_{\vec{n}} = \bigcup_{k\in\omega} D^{\vec{n}}_k$,

(b) (i) $E^{\vec{n}}_k = \begin{cases} C_{[2k+1,2k+2)} & \text{if } \langle k, k+1\rangle \in \text{Even}^{\vec{n}} \times \text{Even}^{\vec{n}}, \\ C_{[2k+1,2k+2]} & \text{if } \langle k, k+1\rangle \in \text{Even}^{\vec{n}} \times \text{Odd}^{\vec{n}}, \\ C_{(2k+1,2k+2)} & \text{if } \langle k, k+1\rangle \in \text{Odd}^{\vec{n}} \times \text{Even}^{\vec{n}}, \\ C_{(2k+1,2k+2]} & \text{if } \langle k, k+1\rangle \in \text{Odd}^{\vec{n}} \times \text{Odd}^{\vec{n}}, \end{cases}$

(ii) $E_{\vec{n}} = \bigcup_{k\in\omega} E^{\vec{n}}_k$.

Lemma 2.5. *Suppose that $\vec{n} = \langle n_i \mid i \in \omega\rangle$ is a strictly increasing sequence of natural numbers with $n_0 = 0$. Then $\langle D_{\vec{n}}, E_{\vec{n}}\rangle$ is a partition of X_0.*

Proof. This follows from Lemma 2.2. □

Remark 2.6. There is a metric space (X, d), x^*, r as above, strictly increasing sequences $\vec{m}$, $\vec{n}$ of natural numbers and a continuous reduction F of $D_{\vec{m}}$ to $D_{\vec{n}}$ such that the subsets $D^{\vec{m}}_k$ and $D^{\vec{n}}_l$ do not match via F. The following arguments shows that the sets $D^{\vec{m}}_k$ and $D^{\vec{n}}_l$ approximately match in a well-defined sense.

The following definition is used to code information about the sets $D_{\vec{n}}$ and about continuous reductions between them in graphs, graph colorings and maps between graphs.

Definition 2.7. (a) An *(undirected, symmetric) graph* $G = \langle V, E\rangle$ consists of a set of vertices $V = V(G)$ and a set of edges $E = E(G) \subseteq V^2$ such that for all $\langle v, w\rangle \in V^2$, $\langle v, w\rangle \in E$ if and only if $\langle w, v\rangle \in E$. Since all graphs are symmetric, we will identify $\langle v, w\rangle$ with $\langle w, v\rangle$.[2]

(b) A *grid* $H = \langle G, c\rangle$ consists of a graph $G = G(H)$ and a pair $c = c(H) = c^H = \langle c^V, c^E\rangle$, where $V = V(H) = V(G)$, $E = E(H) = E(G)$, $V_0 = V_0(H) = V_0(G) \subseteq V$, $c^V : V_0 \to 2$ and $c^E : E \to 2$ satisfy the following conditions.
   (i) $|V| \geq 2$.
   (ii) $V \subseteq \mathbb{Z}$ is *connected* in $\mathbb{Z}$, i.e. $k, l, m \in \mathbb{Z}$, $k \leq l \leq m$ and $k, m \in V$ imply that $l \in V$.
   (iii) V_0 consists of the elements of V without maximal and minimal elements.
   (iv) $E = \{\langle k, l\rangle \in V^2 \mid |k - l| = 1\}$.
   (v) If $n \in V$ is not maximal and not minimal in V and $i < 2$, then there is some $k \in V$ with $\langle n, k\rangle \in E(H)$ and $c^E(\langle n, k\rangle) = i$.[3]

   We will identify H with $V(H) \cup E(H)$ and c^H with $c^V \cup c^E$.

(c) Suppose that H, I are grids. A *reduction* $f : H \to I$ is a pair $f = \langle f^V, f^E\rangle$, where $V = V(H)$, $E = E(H)$, $f^V : V_0(H) \to V_0(I)$, $f^E : E(H) \to E(I)$ such that the following conditions hold for all $v, w \in V(H)$.
   (i) If $v \in V_0(H)$, then $c^H(v) = c^I(f(v))$.
   (ii) If $v \in V_0(H)$ and $\langle v, w\rangle \in E(H)$, then there is some $u \in V(I)$ with $f^E(\langle v, w\rangle) = \langle f^V(v), u\rangle$.
   (iii) If $\langle v, w\rangle \in E(H)$, then $c^H(\langle v, w\rangle) = c^I(\langle f^V(v), f^V(w)\rangle)$.

   We will identify f with $f^V \cup f^E$.

(d) Suppose that H and $\bar{H}$ are grids. An *unfolding* of $\langle H, \bar{H}\rangle$ is a triple $\langle I, f, \bar{f}\rangle$, where I is a finite grid and $f : I \to H$, $\bar{f} : I \to \bar{H}$ are reductions.

Lemma 2.8. *Suppose that $f : H \to I$ is a grid reduction. Then for all $v, w \in V_0(H)$ with $\langle v, w\rangle \in E(H)$, $f(\langle v, w\rangle) = \langle f(v), f(w)\rangle$ if and only if $f(v) \neq f(w)$.*

Proof. This follows from Definition 2.7 (c) (ii) applied to $\langle v, w\rangle$ and $\langle w, v\rangle$. □

The followings sets code information about the sets $D^{\vec{n}}_k$ and $E^{\vec{n}}_k$.

Definition 2.9. Suppose that $\vec{n} = \langle n_i \mid i \in \omega\rangle$ is a strictly increasing sequence of natural numbers with $n_0 = 0$ and $k \in \omega$. We define the following sets.

(a) (i) $D^{\vec{n},\text{edge}}_k = \{\langle 2k, 2k+1\rangle, \langle 2k+1, 2k\rangle\}$,

[2] For instance, for reductions f as defined in Definition 2.7 (c), we will assume that $f(\langle v, w\rangle) = f(\langle w, v\rangle) = \langle s, t\rangle = \langle t, s\rangle$.

[3] This condition avoids additional cases in the following proofs.

(ii) $D^{\vec{n},\text{edge}} = \bigcup_{k\in\omega} D^{\vec{n},\text{edge}}_k$,

(iii) $D^{\vec{n},\text{vertex}}_k = \begin{cases} \{2k\} \text{ if } k \in \text{Even}^{\vec{n}}, \\ \{2k+1\} \text{ if } k \in \text{Odd}^{\vec{n}}, \end{cases}$

(iv) $D^{\vec{n},\text{vertex}} = \bigcup_{k\in\omega} D^{\vec{n},\text{vertex}}_k$,

(b) (i) $E^{\vec{n},\text{edge}}_k = \{\langle 2k+1, 2k+2\rangle, \langle 2k+2, 2k+1\rangle\}$,

(ii) $E^{\vec{n},\text{edge}} = \bigcup_{k\in\omega} E^{\vec{n},\text{edge}}_k$,

(iii) $E^{\vec{n},\text{vertex}}_k = \begin{cases} \{2k+1\} & \text{if } \langle k, k+1\rangle \in \text{Even}^{\vec{n}} \times \text{Even}^{\vec{n}}, \\ \{2k+1, 2k+2\} & \text{if } \langle k, k+1\rangle \in \text{Even}^{\vec{n}} \times \text{Odd}^{\vec{n}}, \\ \emptyset & \text{if } \langle k, k+1\rangle \in \text{Odd}^{\vec{n}} \times \text{Even}^{\vec{n}}, \\ \{2k+2\} & \text{if } \langle k, k+1\rangle \in \text{Odd}^{\vec{n}} \times \text{Odd}^{\vec{n}}, \end{cases}$

(iv) $E^{\vec{n},\text{vertex}} = \bigcup_{k\in\omega} E^{\vec{n},\text{vertex}}_k$.

The following grid $H_{\vec{n}}$ codes information about the sets $D^{\vec{n}}_k$ and $E^{\vec{n}}_k$.

Definition 2.10. Suppose that $\vec{n} = \langle n_i \mid i \in \omega\rangle$ is a strictly increasing sequence of natural numbers with $n_0 = 0$. Let $H_{\vec{n}} = H = \langle G_{\vec{n}}, c_{\vec{n}}\rangle = \langle G, c\rangle$ denote the unique infinite grid with the following properties.

(a) $V = V(H) = \omega$,
(b) $c(\langle 2k, 2k+1\rangle) = 1$ and $c(\langle 2k+1, 2k+2\rangle) = 0$ for all $k \in V$,
(c) $c(2k) = 1$ and $c(2k+1) = 0$ if $k \in \text{Even}^{\vec{n}}$,
(d) $c(2k) = 0$ and $c(2k+1) = 1$ if $k \in \text{Odd}^{\vec{n}}$.

We now fix some notation for the following proofs. Suppose that $\vec{m} = \langle m_i \mid i \in \omega\rangle$, $\vec{n} = \langle n_i \mid i \in \omega\rangle$ are strictly increasing sequences of natural numbers with $m_0 = 0$ and $n_0 = 0$ and $F\colon X \to X$ is a continuous reduction of $D_{\vec{m}}$ to $D_{\vec{n}}$. Since $D_{\vec{m}}, D_{\vec{n}} \subseteq X_0$ and $x^* \in D_{\vec{m}}$, this implies that $F(x^*) \in X_0$.

In the next definition, we consider the elements x of X_0 such that the pair $\langle x, F(x)\rangle$ is consistent with a given unfolding as defined in Definition 2.7 (d).

Definition 2.11. Suppose that $\xi = \langle I, f, \bar{f}\rangle$ is an unfolding of $\langle H_{\vec{m}}, H_{\vec{n}}\rangle$, $f\colon I \to H_{\vec{m}}$, $\bar{f}\colon I \to H_{\vec{n}}$ and $n, n_0 \in \omega$. An element x of X_0 is *compatible* with ξ if the following conditions hold.

(a) If $x \in D^{\vec{m}}_k$, then there is some l and some $e \in E(I)$ with $F(x) \in D^{\vec{n}}_l$, $f(e) \in D^{\vec{m},\text{edge}}_k$ and $\bar{f}(e) \in D^{\vec{n},\text{edge}}_l$.
(b) If $x \in E^{\vec{m}}_k$, then there is some l and some $e \in E(I)$ with $F(x) \in E^{\vec{n}}_l$, $f(e) \in E^{\vec{m},\text{edge}}_k$ and $\bar{f}(e) \in E^{\vec{n},\text{edge}}_l$.
(c) n_0 is a *base point* for $\langle \xi, n\rangle$ if $f(n_0) = 0$ and $\bar{f}(n_0) = n$.

Definition 2.12. Suppose that ξ is an unfolding of $\langle H_{\vec{m}}, H_{\vec{n}}\rangle$.

(a) Let M_ξ denote the set of $x \in X_0$ that are compatible with ξ.
(b) Let M_n denote the union of all sets M_ζ, where ζ is any unfolding of $\langle H_{\vec{m}}, H_{\vec{n}}\rangle$ with a base point for $\langle \zeta, n\rangle$.

Definition 2.13. Suppose that $x \in U \subseteq X_0$. The set U is *small at x* if the following conditions hold for $\langle \bar{x}, \bar{U}\rangle = \langle x, U\rangle$, $\langle \bar{x}, \bar{U}\rangle = \langle F(x), F[U]\rangle$ and all $n \in \omega$.

(a) If $d(x^*, \bar{x}) < r_n$, then $U \subseteq C_{<n}$.
(b) If $d(x^*, \bar{x}) = r_n$ and $m < n$, then $U \subseteq C_{>m}$.
(c) If $d(x^*, \bar{x}) = r_n$ and $m > n$, then $U \subseteq C_{<m}$.
(d) If $d(x^*, \bar{x}) > r_n$, then $U \subseteq C_{>n}$.

Lemma 2.14. *M_n is an open subset of X_0 for all $n \in \omega$.*

Proof. Suppose that $x \in M_n$. Suppose that $\xi = \langle I, f, \bar{f}\rangle$ is an unfolding of $\langle H_{\vec{m}}, H_{\vec{n}}\rangle$ with a base point n_0 for n. Since F is continuous, there is an open subset U of X_0 with $x \in U$ that is small at x. We will prove that $U \subseteq M_n$. Suppose that $y \in U$. We will use that F is a reduction of $D_{\vec{m}}$ to $D_{\vec{n}}$.

In the following arguments, we will write that a case is *analogous* to another case if the cases are symmetric and the same subcases appear in the proof, possibly with different indices. We will write that a case is *similar* to another case if the proof has the same steps, possibly in a different order, with different subcases or a different number of subcases. We will give one proof of each type and omit similar cases.

The unfolding ξ is extended by adding elements to I above $\max(I)$ or below $\min(I)$ in Case 1.1.2.4, Case 1.1.2.5, Case 2.2.1.1.3.1 and analogous cases. The unfolding ξ is extended by adding elements to I between elements of I in Case 2.2.1.1.2.1.2 and analogous cases.

Case 1. $\langle x, F(x)\rangle \in D_k^{\vec{m}} \times D_l^{\vec{n}}$ for some k, l.

Suppose that $e \in E(I)$ with $f(e) = \langle 2k, 2k+1\rangle$ and $\bar{f}(e) = \langle 2l, 2l+1\rangle$.

Case 1.1. $\langle k, l\rangle \in \mathrm{Even}^{\vec{m}} \times \mathrm{Even}^{\vec{n}}$.

Case 1.1.1. $y \in D_k^{\vec{m}}$.

Since $F(x) \in D_l^{\vec{n}}$, F is a reduction of $D_{\vec{m}}$ to $D_{\vec{n}}$ and $y \in U$, we have $F(y) \in D_l^{\vec{n}}$. Then $y \in M_\xi$.

Case 1.1.2. $y \in E_{\bar{k}}^{\vec{m}}$ for some $\bar{k} \in \omega$.

Since $k \in \mathrm{Even}^{\vec{m}}$ and $y \in U$, we have $k > 0$ and $k = \bar{k}+1$. Since $l \in \mathrm{Even}^{\vec{n}}$ and $y \in U$, we have $l > 0$, $l = \bar{l}+1$ and $F(y) \in E_{\bar{l}}^{\vec{n}}$.

Suppose that $e = \langle i, i+1\rangle$.

Case 1.1.2.1. $i, i+1 \in V_0(I)$, $c^I(i) = 0$ and $c^I(i+1) = 0$.

This contradicts the assumption that f is a reduction of I to $H_{\vec{m}}$.

Case 1.1.2.2. $i \in V_0(I)$ and $c^I(i) = 1$.

Since f is a reduction of I to $H_{\vec{m}}$, we have $f(\langle i-1, i\rangle) = \langle 2\bar{k}+1, 2k\rangle$. Since $\bar{f}$ is a reduction of I to $H_{\vec{n}}$, we have $\bar{f}(\langle i-1, i\rangle) = \langle 2\bar{l}+1, 2l\rangle$. Then $y \in M_\xi$.

Case 1.1.2.3.[4] $i+1 \in V_0(I)$ and $c^I(i+1) = 1$.

Since f is a reduction of I to $H_{\vec{m}}$, we have $f(\langle i, i+1\rangle) = \langle 2\bar{k}+1, 2k\rangle$. Since $\bar{f}$ is a reduction of I to $H_{\vec{n}}$, we have $\bar{f}(\langle i, i+1\rangle) = \langle 2\bar{l}+1, 2l\rangle$. Then $y \in M_\xi$.

Case 1.1.2.4. $i \notin V_0(I)$.

Let $V(\bar{I}) = V(I) \cup \{i\}$, $E(\bar{I}) = E(I) \cup \{\langle i, i-1\rangle, \langle i-1, i\rangle\}$, $c^{\bar{I}}(v) = c^I(v)$ for $v \in V_0(I)$, $c^{\bar{I}}(u) = c^I(u)$ for $u \in E(I)$, $c^{\bar{I}}(i) = 1$, $c^{\bar{I}}(\langle i-1, i\rangle) = 0$. Let $g\colon \bar{I} \to H_{\vec{m}}$, $g(v) = f(v)$ for $v \in V_0(I) \cup E(I)$, $g(i) = 2k$, $g(\langle i-1, i\rangle) = \langle 2\bar{k}+1, 2k\rangle$. Let $\bar{g}\colon \bar{I} \to H_{\vec{n}}$, $\bar{g}(v) = \bar{f}(v)$ for $v \in V_0(I) \cup E(I)$, $\bar{g}(i) = 2l$, $\bar{g}(\langle i-1, i\rangle) = \langle 2\bar{l}+2, 2l\rangle$. Let $\zeta = \langle \bar{I}, g, \bar{g}\rangle$. Then ζ has a base point for n and $y \in M_\zeta$.

Case 1.1.2.5. $i+1 \notin V_0(I)$.

Let $V(\bar{I}) = V(I) \cup \{i+1\}$, $E(\bar{I}) = E(I) \cup \{\langle i+1, i+2\rangle, \langle i+2, i+1\rangle\}$, $c^{\bar{I}}(v) = c^I(v)$ for $v \in V_0(I)$, $c^{\bar{I}}(u) = c^I(u)$ for $u \in E(I)$, $c^{\bar{I}}(i+1) = 1$, $c^{\bar{I}}(\langle i+1, i+2\rangle) = 0$. Let $g\colon \bar{I} \to H_{\vec{m}}$, $g(v) = f(v)$ for $v \in V_0(I) \cup E(I)$, $g(i+1) = 2k$, $g(\langle i+1, i+2\rangle) = \langle 2\bar{k}+1, 2k\rangle$. Let $\bar{g}\colon \bar{I} \to H_{\vec{n}}$, $\bar{g}(v) = \bar{f}(v)$ for $v \in V_0(I) \cup E(I)$, $\bar{g}(i+1) = 2l$, $\bar{g}(\langle i+1, i+2\rangle) = \langle 2\bar{l}+1, 2l\rangle$. Let $\zeta = \langle \bar{I}, g, \bar{g}\rangle$. Then ζ has a base point for n and $y \in M_\zeta$.

Case 1.2. $\langle k, l\rangle \in \mathrm{Even}^{\vec{m}} \times \mathrm{Odd}^{\vec{n}}$.

This is analogous to Case 1.1.

Case 1.3. $\langle k, l\rangle \in \mathrm{Odd}^{\vec{m}} \times \mathrm{Even}^{\vec{n}}$.

This is analogous to Case 1.1.

Case 1.4. $\langle k, l\rangle \in \mathrm{Odd}^{\vec{m}} \times \mathrm{Odd}^{\vec{n}}$.

This is analogous to Case 1.1.

Case 2. $\langle x, F(x)\rangle \in E_k^{\vec{m}} \times E_l^{\vec{n}}$ for some k, l.

Suppose that $e \in E(I)$ with $f(e) = \langle 2k+1, 2k+2\rangle$ and $\bar{f}(e) = \langle 2l+1, 2l+2\rangle$.

Case 2.1. $\langle k, k+1, l, l+1\rangle \in \mathrm{Even}^{\vec{m}} \times \mathrm{Even}^{\vec{m}} \times \mathrm{Even}^{\vec{n}} \times \mathrm{Even}^{\vec{n}}$.

This is analogous to Case 1.1.

Case 2.2. $\langle k, k+1, l, l+1\rangle \in \mathrm{Even}^{\vec{m}} \times \mathrm{Even}^{\vec{m}} \times \mathrm{Even}^{\vec{n}} \times \mathrm{Odd}^{\vec{n}}$.

Case 2.2.1. $y \in D_k^{\vec{m}}$.

Case 2.2.1.1. $f(i) = 2k+1$ and $f(i+1) = 2k+2$.

Case 2.2.1.1.1. $i \in V_0(I)$ and $c^I(i) = 1$.

This contradicts the assumption that f is a reduction of I to $H_{\vec{m}}$.

[4]This and the previous case (and some of the following cases) are not exclusive and either may be chosen for the proof.

Case 2.2.1.1.2. $i \in V_0(I)$ and $c^I(i) = 0$.

Since f is a reduction of I to $H_{\vec{m}}$, we have $f(\langle i-1, i\rangle) = \langle 2k, 2k+1\rangle$. Since $\bar{f}$ is a reduction of I to $H_{\vec{n}}$, we have $\bar{f}(\langle i-1, i\rangle) = \langle 2l, 2l+1\rangle$ or $\bar{f}(\langle i-1, i\rangle) = \langle 2l+2, 2l+3\rangle$.

Case 2.2.1.1.2.1. $\bar{f}(\langle i-1, i\rangle) = \langle 2l, 2l+1\rangle$.

Case 2.2.1.1.2.1.1. $F(y) \in D_l^{\vec{n}}$.

Then $y \in M_\xi$.

Case 2.2.1.1.2.1.2. $F(y) \in D_{l+1}^{\vec{n}}$.

Suppose that $r, r+1, \dots < i-1 < i < i+1, \dots, s$ enumerate $V(I)$ in increasing order. The proof for the subcases $r = i-1$ and $s = i+1$ is analogous and is omitted.

We define $\zeta = \langle \bar{I}, g, \bar{g}\rangle$ as follows. Suppose that $r-2, r-1, \dots, s$ is the enumeration of $V(\bar{I})$ in increasing order.

Let $g(j) = f(j+2)$ for all j with $r-2 < j \leq i-3$. Let $g(i-2) = 2k+1 = g(i-1) = 2k+1$. Let $g(j) = f(j)$ for all j with $i \leq j < s$. There is a unique consistent extension of g from $V(\bar{I})$ to $E(\bar{I})$ by the definition of $H_{\vec{m}}$.

Let $\bar{g}(j) = \bar{f}(j+2)$ for all j with $r-2 < j \leq i-3$. Let $\bar{g}(i-2) = 2l+2$ and $\bar{g}(i-1) = 2l+1$. Let $\bar{g}(j) = \bar{f}(j)$ for all j with $i \leq j < s$. There is a unique consistent extension of $\bar{g}$ from $V(\bar{I})$ to $E(\bar{I})$ by the definition of $H_{\vec{n}}$.

Let $c^{\bar{I}}(j) = c(j+2)$ for all j with $r-2 < j \leq i-3$. Let $c^{\bar{I}}(i-2) = 0 = c^{\bar{I}}(i-1) = 0$. Let $c^{\bar{I}}(j) = c(j)$ for all j with $i \leq j < s$.

Let $c^{\bar{I}}(\langle j, j+1\rangle) = c(\langle j+2, j+3\rangle)$ for all j with $r-2 < j \leq i-3$. Let $c^{\bar{I}}(\langle i-2, i-1\rangle) = 0$ and $c^{\bar{I}}(\langle i-1, i\rangle) = 1$. Let $c^{\bar{I}}(\langle j, j+1\rangle) = c(\langle j, j+1\rangle)$ for all j with $i \leq j < s$.

Then ζ has a base point for n and $y \in M_\zeta$.

Case 2.2.1.1.2.2. $\bar{f}(\langle i-1, i\rangle) = \langle 2l+2, 2l+3\rangle$.

This is analogous to Case 2.2.1.1.2.1.

Case 2.2.1.1.3. $i \notin V_0(I)$.

Case 2.2.1.1.3.1. $F(y) \in D_l^{\vec{n}}$.

Suppose that $i, i+1, \dots, s$ is the order preserving enumeration of $V(I)$. We define $\zeta = \langle \bar{I}, g, \bar{g}\rangle$ as follows. Let $V(\bar{I}) = I \cup \{i-1\}$. Let $c^{\bar{I}}(i) = 0$. Let $g(i-1) = 2k$ and $g(j) = f(j)$ for $j \in I$. Let $g(\langle i-1, i\rangle) = \langle 2k, 2k+1\rangle$ and $g(\langle j, j+1\rangle) = f(\langle j, j+1\rangle)$ for $i \leq j < s$. Let $\bar{g}(i-1) = 2l$ and $\bar{g}(j) = \bar{f}(j)$ for $i \leq j < s$. Let $\bar{g}(\langle i-1, i\rangle) = \langle 2k, 2k+1\rangle$ and $\bar{f}(\langle j, j+1\rangle) = \bar{f}(\langle j, j+1\rangle)$ for $i \leq j < s$.

Then ζ has a base point for n and $y \in M_\zeta$.

Case 2.2.1.1.3.2. $F(y) \in D_{l+1}^{\vec{n}}$.

This is analogous to Case 2.2.1.1.3.1.

Case 2.2.1.2. $f(i) = 2k+2$ and $f(i+1) = 2k+1$.

This is analogous to Case 2.2.1.1.

Case 2.2.2. $y \in E_k^{\vec{m}}$. Since $F(x) \in E_l^{\vec{n}}$, F is a reduction of $D_{\vec{m}}$ to $D_{\vec{n}}$ and $y \in U$, we have $F(y) \in E_l^{\vec{n}}$. Then $y \in M_\xi$.

Suppose that

Case 2.3. $\langle k, k+1, l, l+1\rangle \in \mathrm{Even}^{\vec{m}} \times \mathrm{Even}^{\vec{m}} \times \mathrm{Odd}^{\vec{n}} \times \mathrm{Even}^{\vec{n}}$.

Case 2.3.1. $y \in E_k^{\vec{m}}$.

Since $F(x) \in E_l^{\vec{n}}$, F is a reduction of $D_{\vec{m}}$ to $D_{\vec{n}}$ and $y \in U$, we have $F(y) \in E_l^{\vec{n}}$. Then $y \in M_\xi$.

Case 2.3.2. $y \in D_k^{\vec{m}}$.

Since $F(x) \in E_l^{\vec{n}}$, F is a reduction of $D_{\vec{m}}$ to $D_{\vec{n}}$ and $y \in U$, we have $F(y) \in D_l^{\vec{n}}$ or $F(y) \in D_{l+1}^{\vec{n}}$. Since $y \in U$ and $\langle l, l+1\rangle \in \mathrm{Odd}^{\vec{m}} \times \mathrm{Even}^{\vec{n}}$, $F(y) \in E_l^{\vec{n}}$. This contradicts the previous property of $F(y)$.

Case 2.4. $\langle k, k+1, l, l+1\rangle \in \mathrm{Even}^{\vec{m}} \times \mathrm{Even}^{\vec{m}} \times \mathrm{Odd}^{\vec{n}} \times \mathrm{Odd}^{\vec{n}}$.

This is analogous to Case 1.1.

Case 2.5. $\langle k, k+1, l, l+1\rangle \in \mathrm{Even}^{\vec{m}} \times \mathrm{Odd}^{\vec{m}} \times \mathrm{Even}^{\vec{n}} \times \mathrm{Even}^{\vec{n}}$.

This is similar to Case 1.1.

Case 2.6. $\langle k, k+1, l, l+1\rangle \in \mathrm{Even}^{\vec{m}} \times \mathrm{Odd}^{\vec{m}} \times \mathrm{Even}^{\vec{n}} \times \mathrm{Odd}^{\vec{n}}$.

This is similar to Case 2.2.

Case 2.7. $\langle k, k+1, l, l+1\rangle \in \mathrm{Even}^{\vec{m}} \times \mathrm{Odd}^{\vec{m}} \times \mathrm{Odd}^{\vec{n}} \times \mathrm{Even}^{\vec{n}}$.

This is similar to Case 2.3.

Case 2.8. $\langle k, k+1, l, l+1\rangle \in \mathrm{Even}^{\vec{m}} \times \mathrm{Odd}^{\vec{m}} \times \mathrm{Odd}^{\vec{n}} \times \mathrm{Odd}^{\vec{n}}$.

This is analogous to Case 2.5.

Case 2.9. $\langle k, k+1, l, l+1\rangle \in \mathrm{Odd}^{\vec{m}} \times \mathrm{Even}^{\vec{m}} \times \mathrm{Even}^{\vec{n}} \times \mathrm{Even}^{\vec{n}}$.

Since $y \in U$ and $\langle k, k+1\rangle \in \mathrm{Odd}^{\vec{m}} \times \mathrm{Even}^{\vec{m}}$, we have $y \in E^{\vec{m}}_k$. Since $F(x) \in E^{\vec{n}}_l$, F is a reduction of $D_{\vec{m}}$ to $D_{\vec{n}}$ and $y \in U$, we have $F(y) \in E^{\vec{n}}_l$. Then $y \in M_\xi$.

Case 2.10. $\langle k, k+1, l, l+1\rangle \in \mathrm{Odd}^{\vec{m}} \times \mathrm{Even}^{\vec{m}} \times \mathrm{Even}^{\vec{n}} \times \mathrm{Odd}^{\vec{n}}$.

This is similar to Case 2.9.

Case 2.11. $\langle k, k+1, l, l+1\rangle \in \mathrm{Odd}^{\vec{m}} \times \mathrm{Even}^{\vec{m}} \times \mathrm{Odd}^{\vec{n}} \times \mathrm{Even}^{\vec{n}}$.

This is similar to Case 2.9.

Case 2.12. $\langle k, k+1, l, l+1\rangle \in \mathrm{Odd}^{\vec{m}} \times \mathrm{Even}^{\vec{m}} \times \mathrm{Odd}^{\vec{n}} \times \mathrm{Odd}^{\vec{n}}$.

This is similar to Case 2.9.

Case 2.13. $\langle k, k+1, l, l+1\rangle \in \mathrm{Odd}^{\vec{m}} \times \mathrm{Odd}^{\vec{m}} \times \mathrm{Even}^{\vec{n}} \times \mathrm{Even}^{\vec{n}}$.

This is analogous to Case 1.1.

Case 2.14. $\langle k, k+1, l, l+1\rangle \in \mathrm{Odd}^{\vec{m}} \times \mathrm{Odd}^{\vec{m}} \times \mathrm{Even}^{\vec{n}} \times \mathrm{Odd}^{\vec{n}}$.

This is analogous to Case 2.2.

Case 2.15. $\langle k, k+1, l, l+1\rangle \in \mathrm{Odd}^{\vec{m}} \times \mathrm{Odd}^{\vec{m}} \times \mathrm{Odd}^{\vec{n}} \times \mathrm{Even}^{\vec{n}}$.

This is analogous to Case 2.3.

Case 2.16. $\langle k, k+1, l, l+1\rangle \in \mathrm{Odd}^{\vec{m}} \times \mathrm{Odd}^{\vec{m}} \times \mathrm{Odd}^{\vec{n}} \times \mathrm{Odd}^{\vec{n}}$.

This is analogous to Case 1.1. □

Lemma 2.15. *Suppose that $n \in \omega$. Then M_n is a closed subset of X_0.*[5]

Proof. Suppose that $\vec{x} = \langle x_i \mid i \in \omega\rangle$ is a sequence in X_0 with $\lim_i x_i = x$ and $x \in X_0$. Since F is continuous, there is an open subset U of X_0 with $x \in U$ that is small at x. We can assume that $x_i \in U$ for all $i \in \omega$ by replacing $\vec{x}$ with an infinite subsequence.

We can assume that $x_i \in D^{\vec{n}}_k$ if and only if $x_j \in D^{\vec{n}}_k$ for all $i, j, k \in \omega$ by replacing $\vec{x}$ with an infinite subsequence. We can assume that $x_i \in E^{\vec{n}}_k$ if and only if $x_j \in E^{\vec{n}}_k$ for all $i, j, k \in \omega$ by replacing $\vec{x}$ with an infinite subsequence. Suppose that $x_i \in M_n$ for all $i \in \omega$. In particular, $x_i \in X_0$ for all $i \in \omega$. Since F is continuous, we can assume that $F(x_i) \in D^{\vec{n}}_k$ if and only if $F(x_j) \in D^{\vec{n}}_k$ for all $i, j, k \in \omega$ by replacing $\vec{x}$ with an infinite subsequence. Since F is continuous, we can assume that $F(x_i) \in E^{\vec{n}}_k$ if and only if $F(x_j) \in E^{\vec{n}}_k$ for all $i, j, k \in \omega$ by replacing $\vec{x}$ with an infinite subsequence.

Suppose that $\xi = \langle I, f, \bar{f}\rangle$ is an unfolding of $\langle H_{\vec{m}}, H_{\vec{n}}\rangle$ with a base point n_0 for n such that x_0 is compatible with ξ. Then x_i is compatible with ξ for all $i \in \omega$. We will prove that $x \in M_n$.

In the following arguments, we will write that a case is *analogous* to another case if the cases are symmetric and exactly the same subcases appear in the proof, possibly with different indices. We will write that a case is *similar* to another case if the proof has the same steps, possibly in a different order, different subcases or a different number of subcases. We will give one proof of each type and omit similar cases.

The unfolding ξ is extended by adding elements to I above $\max(I)$ or below $\min(I)$ in Case 1.1.2.4, Case 1.1.2.5, Case 2.3.1.1.3.1 and analogous cases. The unfolding ξ is extended by adding elements to I between elements of I in Case 2.3.1.1.2.1.2 and analogous cases.

Case 1. $\langle x_0, F(x_0)\rangle \in D^{\vec{m}}_k \times D^{\vec{n}}_l$ for some k, l.

Suppose that $e \in E(I)$ with $f(e) = \langle 2k, 2k+1\rangle$ and $\bar{f}(e) = \langle 2l, 2l+1\rangle$.

Case 1.1. $\langle k, l\rangle \in \mathrm{Even}^{\vec{m}} \times \mathrm{Even}^{\vec{n}}$.

Case 1.1.1. $x \in D^{\vec{m}}_k$.

Since $F(x_0) \in D^{\vec{n}}_l$, F is a reduction of $D_{\vec{m}}$ to $D_{\vec{n}}$ and $x \in U$, we have $F(x) \in D^{\vec{n}}_l$. Then $x \in M_\xi$.

Case 1.1.2. $x \in E^{\vec{m}}_{\bar{k}}$ for some $\bar{k} \in \omega$.

Since $k \in \mathrm{Even}^{\vec{m}}$, we have $k = \bar{k}$. Since $l \in \mathrm{Even}^{\vec{n}}$, we have $F(x) \in E^{\vec{n}}_l$.

Suppose that $e = \langle i, i+1\rangle$.

Case 1.1.2.1. $i, i+1 \in V_0(I)$, $c^I(i) = 0$ and $c^I(i+1) = 0$.

This contradicts the assumption that f is a reduction of I to $H_{\vec{m}}$.

Case 1.1.2.2. $i \in V_0(I)$ and $c^I(i) = 1$.

[5] M_n is not necessarily closed in X.

Since f is a reduction of I to $H_{\vec{m}}$, we have $f(\langle i-1,i\rangle)=\langle 2k+1,2k+2\rangle$. Since $\bar{f}$ is a reduction of I to $H_{\vec{n}}$, we have $\bar{f}(\langle i-1,i\rangle)=\langle 2l+1,2l+2\rangle$. Then $x\in M_\xi$.

Case 1.1.2.3. $i+1\in V_0(I)$ and $c^I(i+1)=1$.

Since f is a reduction of I to $H_{\vec{m}}$, we have $f(\langle i,i+1\rangle)=\langle 2k+1,2k+2\rangle$. Since $\bar{f}$ is a reduction of I to $H_{\vec{n}}$, we have $\bar{f}(\langle i,i+1\rangle)=\langle 2l+1,2l+2\rangle$. Then $x\in M_\xi$.

Case 1.1.2.4. $i\notin V_0(I)$.

Let $V(\bar{I})=V(I)\cup\{i\}$, $E(\bar{I})=E(I)\cup\{\langle i,i-1\rangle,\langle i-1,i\rangle\}$, $c^{\bar{I}}(v)=c^I(v)$ for $v\in V_0(I)$, $c^{\bar{I}}(u)=c^I(u)$ for $u\in E(I)$, $c^{\bar{I}}(i)=0$, $c^{\bar{I}}(\langle i-1,i\rangle)=0$. Let $g\colon \bar{I}\to H_{\vec{m}}$, $g(v)=f(v)$ for $v\in V_0(I)\cup E(I)$, $g(i)=2k+1$, $g(\langle i-1,i\rangle)=\langle 2k+1,2k+2\rangle$. Let $\bar{g}\colon \bar{I}\to H_{\vec{n}}$, $\bar{g}(v)=\bar{f}(v)$ for $v\in V_0(I)\cup E(I)$, $\bar{g}(i)=2l+1$, $\bar{g}(\langle i-1,i\rangle)=\langle 2l+1,2l+2\rangle$. Let $\zeta=\langle \bar{I},g,\bar{g}\rangle$. Then ζ has a base point for n and $x\in M_\zeta$.

Case 1.1.2.5. $i+1\notin V_0(I)$.

Let $V(\bar{I})=V(I)\cup\{i+1\}$, $E(\bar{I})=E(I)\cup\{\langle i+1,i+2\rangle,\langle i+2,i+1\rangle\}$, $c^{\bar{I}}(v)=c^I(v)$ for $v\in V_0(I)$, $c^{\bar{I}}(u)=c^I(u)$ for $u\in E(I)$, $c^{\bar{I}}(i+1)=0$, $c^{\bar{I}}(\langle i+1,i+2\rangle)=0$. Let $g\colon \bar{I}\to H_{\vec{m}}$, $g(v)=f(v)$ for $v\in V_0(I)\cup E(I)$, $g(i+1)=2k+1$, $g(\langle i+1,i+2\rangle)=\langle 2k+1,2k+2\rangle$. Let $\bar{g}\colon \bar{I}\to H_{\vec{n}}$, $\bar{g}(v)=\bar{f}(v)$ for $v\in V_0(I)\cup E(I)$, $\bar{g}(i+1)=2l+1$, $\bar{g}(\langle i+1,i+2\rangle)=\langle 2l+1,2l+2\rangle$. Let $\zeta=\langle \bar{I},g,\bar{g}\rangle$. Then ζ has a base point for n and $x\in M_\zeta$.

Case 1.2. $\langle k,l\rangle\in \mathrm{Even}^{\vec{m}}\times \mathrm{Odd}^{\vec{n}}$.

This is analogous to Case 1.1 for $l>0$. It is similar to Case 1.1 for $l=0$.

Case 1.3. $\langle k,l\rangle\in \mathrm{Odd}^{\vec{m}}\times \mathrm{Even}^{\vec{n}}$.

This is analogous to Case 1.1 for $k>0$. It is similar to Case 1.1 for $k=0$.

Case 1.4. $\langle k,l\rangle\in \mathrm{Odd}^{\vec{m}}\times \mathrm{Odd}^{\vec{n}}$.

This is analogous to Case 1.1 for $k,l>0$. It is similar to Case 1.1 for $k=0$ and for $l=0$.

Case 2. $\langle x_0,F(x_0)\rangle\in E^{\vec{m}}_k\times E^{\vec{n}}_l$ for some k,l.

Suppose that $e\in E(I)$ with $f(e)=\langle 2k+1,2k+2\rangle$ and $\bar{f}(e)=\langle 2l+1,2l+2\rangle$.

Case 2.1. $\langle k,k+1,l,l+1\rangle\in \mathrm{Even}^{\vec{m}}\times \mathrm{Even}^{\vec{m}}\times \mathrm{Even}^{\vec{n}}\times \mathrm{Even}^{\vec{n}}$.

This is analogous to Case 1.1.

Case 2.2. $\langle k,k+1,l,l+1\rangle\in \mathrm{Even}^{\vec{m}}\times \mathrm{Even}^{\vec{m}}\times \mathrm{Even}^{\vec{n}}\times \mathrm{Odd}^{\vec{n}}$.

Since $F(x_i)\in E^{\vec{n}}_l$ for all $i\in\omega$ and F is continuous, we have $F(x)\in E^{\vec{n}}_l$.

Case 2.2.1. $x\in D^{\vec{m}}_{k+1}$.

The fact that F is a reduction of $D_{\vec{m}}$ to $D_{\vec{n}}$ contradicts the fact that $F(x)\in E^{\vec{n}}_l$.

Case 2.2.2. $x\in E^{\vec{m}}_k$.

Since $F(x)\in E^{\vec{n}}_l$, we have $x\in M_\xi$.

Case 2.3. $\langle k,k+1,l,l+1\rangle\in \mathrm{Even}^{\vec{m}}\times \mathrm{Even}^{\vec{m}}\times \mathrm{Odd}^{\vec{n}}\times \mathrm{Even}^{\vec{n}}$.

Case 2.3.1. $x\in D^{\vec{m}}_{k+1}$.

Suppose that $e=\langle i,i+1\rangle$.

Case 2.3.1.1. $f(i)=2k+1$ and $f(i+1)=2k+2$.

Case 2.3.1.1.1. $i\in V_0(I)$ and $c^I(i)=0$.

This contradicts the assumption that f is a reduction of I to $H_{\vec{m}}$.

Case 2.3.1.1.2. $i\in V_0(I)$ and $c^I(i)=1$.

Since f is a reduction of I to $H_{\vec{m}}$, we have $f(\langle i-1,i\rangle)=\langle 2k+2,2k+3\rangle$. Since $\bar{f}$ is a reduction of I to $H_{\vec{n}}$, we have $\bar{f}(\langle i-1,i\rangle)=\langle 2l,2l+1\rangle$ or $\bar{f}(\langle i-1,i\rangle)=\langle 2l+2,2l+3\rangle$.

Case 2.3.1.1.2.1. $\bar{f}(\langle i-1,i\rangle)=\langle 2l,2l+1\rangle$.

Case 2.3.1.1.2.1.1. $F(x)\in D^{\vec{n}}_l$.

Then $x\in M_\xi$.

Case 2.3.1.1.2.1.2. $F(x)\in D^{\vec{n}}_{l+1}$.

Suppose that $r,r+1,\dots<i-1<i<i+1,\dots,s$ enumerate $V(I)$ in increasing order. The proof for the subcases $r=i-1$ and $s=i+1$ is analogous and is omitted.

We define $\zeta=\langle \bar{I},g,\bar{g}\rangle$ as follows. Suppose that $r-2,r-1,\dots,s$ is the enumeration of $V(\bar{I})$ in increasing order.

Let $g(j)=f(j+2)$ for all j with $r-2<j\le i-3$. Let $g(i-2)=2k+1$ and $g(i-1)=g(i)=2k+2$. Let $g(j)=f(j)$ for all j with $i+1\le j<s$. There is a unique consistent extension of g from $V(\bar{I})$ to $E(\bar{I})$ by the definition of $H_{\vec{m}}$.

Let $\bar{g}(j) = \bar{f}(j+2)$ for all j with $r-2 < j \leq i-2$. Let $\bar{g}(i-1) = \bar{g}(i) = 2l+2$. Let $\bar{g}(j) = \bar{f}(j)$ for all j with $i \leq j < s$. There is a unique consistent extension of $\bar{g}$ from $V(\bar{I})$ to $E(\bar{I})$ by the definition of $H_{\vec{n}}$.

Let $c^{\bar{I}}(j) = c(j+2)$ for all j with $r-2 < j \leq i-2$. Let $c^{\bar{I}}(i-1) = 0 = c^{\bar{I}}(i) = 0$. Let $c^{\bar{I}}(j) = c(j)$ for all j with $i+1 \leq j < s$.

Let $c^{\bar{I}}(\langle j, j+1\rangle) = c(\langle j+2, j+3\rangle)$ for all j with $r-2 < j \leq i-3$. Let $c^{\bar{I}}(\langle i-2, i-1\rangle) = 0$ and $c^{\bar{I}}(\langle i-1, i\rangle) = 1$. Let $c^{\bar{I}}(\langle j, j+1\rangle) = c(\langle j, j+1\rangle)$ for all j with $i \leq j < s$.

Then ζ has a base point for n and $y \in M_\zeta$.

Case 2.3.1.1.2.2. $\bar{f}(\langle i-1, i\rangle) = \langle 2l+2, 2l+3\rangle$.

This is analogous to Case 2.3.1.1.2.1.

Case 2.3.1.1.3. $i \notin V_0(I)$.

Case 2.3.1.1.3.1. $F(y) \in D^{\vec{n}}_l$.

Suppose that $i, i+1, \ldots, s$ is the order preserving enumeration of $V(I)$. We define $\zeta = \langle \bar{I}, g, \bar{g}\rangle$ as follows. Let $V(\bar{I}) = I \cup \{i-1\}$. Let $c^{\bar{I}}(i) = 1$. Let $g(i-1) = 2k+2$ and $g(j) = f(j)$ for $j \in V(I)$. Let $g(\langle i-1, i\rangle) = \langle 2k+2, 2k+3\rangle$ and $g(\langle j, j+1\rangle) = f(\langle j, j+1\rangle)$ for $i \leq j < s$. Let $\bar{g}(i-1) = 2l+2$ and $\bar{g}(j) = \bar{f}(j)$ for $i \leq j < s$. Let $\bar{g}(\langle i-1, i\rangle) = \langle 2k+2, 2k+3\rangle$ and $\bar{g}(\langle j, j+1\rangle) = \bar{f}(\langle j, j+1\rangle)$ for $i \leq j < s$.

Then ζ has a base point for n and $x \in M_\zeta$.

Case 2.3.1.1.3.2. $F(y) \in D^{\vec{n}}_{l+1}$.

This is analogous to Case 2.3.1.1.3.1.

Case 2.3.1.2. $f(i) = 2k+2$ and $f(i+1) = 2k+1$.

This is analogous to Case 2.3.1.1.

Case 2.3.2. $x \in E^{\vec{m}}_k$.

Since F is a reduction of $D_{\vec{m}}$ to $D_{\vec{n}}$, we have $F(x) \in E^{\vec{n}}_l$. Then $x \in M_\xi$.

Case 2.4. $\langle k, k+1, l, l+1\rangle \in \mathrm{Even}^{\vec{m}} \times \mathrm{Even}^{\vec{m}} \times \mathrm{Odd}^{\vec{n}} \times \mathrm{Odd}^{\vec{n}}$.

This is analogous to Case 1.1.

Case 2.5. $\langle k, k+1, l, l+1\rangle \in \mathrm{Even}^{\vec{m}} \times \mathrm{Odd}^{\vec{m}} \times \mathrm{Even}^{\vec{n}} \times \mathrm{Even}^{\vec{n}}$.

Since $\langle k, k+1\rangle \in \mathrm{Even}^{\vec{m}} \times \mathrm{Odd}^{\vec{m}}$, we have $x \in E^{\vec{m}}_k$. Since F is a reduction of $D_{\vec{m}}$ to $D_{\vec{n}}$, we have $F(x) \in E^{\vec{n}}_l$. Then $x \in M_\xi$.

Case 2.6. $\langle k, k+1, l, l+1\rangle \in \mathrm{Even}^{\vec{m}} \times \mathrm{Odd}^{\vec{m}} \times \mathrm{Even}^{\vec{n}} \times \mathrm{Odd}^{\vec{n}}$.

This is analogous to Case 2.5.

Case 2.7. $\langle k, k+1, l, l+1\rangle \in \mathrm{Even}^{\vec{m}} \times \mathrm{Odd}^{\vec{m}} \times \mathrm{Odd}^{\vec{n}} \times \mathrm{Even}^{\vec{n}}$.

This is analogous to Case 2.5.

Case 2.8. $\langle k, k+1, l, l+1\rangle \in \mathrm{Even}^{\vec{m}} \times \mathrm{Odd}^{\vec{m}} \times \mathrm{Odd}^{\vec{n}} \times \mathrm{Odd}^{\vec{n}}$.

This is analogous to Case 2.5.

Case 2.9. $\langle k, k+1, l, l+1\rangle \in \mathrm{Odd}^{\vec{m}} \times \mathrm{Even}^{\vec{m}} \times \mathrm{Even}^{\vec{n}} \times \mathrm{Even}^{\vec{n}}$.

This is similar to case 1.1.

Case 2.10. $\langle k, k+1, l, l+1\rangle \in \mathrm{Odd}^{\vec{m}} \times \mathrm{Even}^{\vec{m}} \times \mathrm{Even}^{\vec{n}} \times \mathrm{Odd}^{\vec{n}}$.

This is similar to Case 2.2.

Case 2.11. $\langle k, k+1, l, l+1\rangle \in \mathrm{Odd}^{\vec{m}} \times \mathrm{Even}^{\vec{m}} \times \mathrm{Odd}^{\vec{n}} \times \mathrm{Even}^{\vec{n}}$.

This is similar to Case 1.1.

Case 2.12. $\langle k, k+1, l, l+1\rangle \in \mathrm{Odd}^{\vec{m}} \times \mathrm{Even}^{\vec{m}} \times \mathrm{Odd}^{\vec{n}} \times \mathrm{Odd}^{\vec{n}}$.

This is similar to Case 2.9.

Case 2.13. $\langle k, k+1, l, l+1\rangle \in \mathrm{Odd}^{\vec{m}} \times \mathrm{Odd}^{\vec{m}} \times \mathrm{Even}^{\vec{n}} \times \mathrm{Even}^{\vec{n}}$.

This is analogous to Case 1.1.

Case 2.14. $\langle k, k+1, l, l+1\rangle \in \mathrm{Odd}^{\vec{m}} \times \mathrm{Odd}^{\vec{m}} \times \mathrm{Even}^{\vec{n}} \times \mathrm{Odd}^{\vec{n}}$.

This is analogous to Case 2.2.

Case 2.15. $\langle k, k+1, l, l+1\rangle \in \mathrm{Odd}^{\vec{m}} \times \mathrm{Odd}^{\vec{m}} \times \mathrm{Odd}^{\vec{n}} \times \mathrm{Even}^{\vec{n}}$.

This is analogous to Case 2.3.

Case 2.16. $\langle k, k+1, l, l+1\rangle \in \mathrm{Odd}^{\vec{m}} \times \mathrm{Odd}^{\vec{m}} \times \mathrm{Odd}^{\vec{n}} \times \mathrm{Odd}^{\vec{n}}$.

This is analogous to Case 1.1. □

Lemma 2.16. *Suppose that $\vec{m} = \langle m_i \mid i \in \omega\rangle$ and $\vec{n} = \langle n_i \mid i \in \omega\rangle$ are sequences of natural numbers with $m_0 = n_0 = 0$ such that the sequences $\vec{m}$, $\vec{n}$, $\langle m_{i+1} - m_i \mid i \in \omega\rangle$ and $\langle n_{i+1} - n_i\rangle$ are strictly increasing. Then there is some $k_0 \in \omega$ such that for every $n \in \omega$, there is some $l_n \in \mathbb{Z}$ with* *the following property. Suppose that k, l are such that $k \geq k_0$, there is an unfolding $\xi = \langle I, f, \bar{f} \rangle$ of $\langle H_{\vec{m}}, H_{\vec{n}} \rangle$ with a base point for n and there is some i with $f(i) = k$ and $\bar{f}(i) = l$. Then $k + l_n = l$.*

Proof. It follows from the assumption that $\vec{m}$, $\vec{n}$, $\langle m_{i+1} - m_i \mid i \in \omega \rangle$ and $\langle n_{i+1} - n_i \rangle$ are strictly increasing, the Definition 2.10 of $H_{\vec{m}}$ and $H_{\vec{n}}$ and the Definition 2.11 of unfolding by a straightforward induction that $k_0 = m_2$ has this property. □

Definition 2.17. Sequences $\vec{m} = \langle m_i \mid i \in \omega \rangle$ and $\vec{n} = \langle n_i \mid i \in \omega \rangle$ of natural numbers are equivalent with respect to the equivalence relation E_{tail} if there is some $i_0 \in \omega$ and some $l \in \mathbb{Z}$ such that for all $i \geq i_0$, $m_i = n_i + l$.

Lemma 2.18. *Suppose that $\vec{m} = \langle m_i \mid i \in \omega \rangle$ and $\vec{n} = \langle n_i \mid i \in \omega \rangle$ are sequences of natural numbers with $m_0 = n_0 = 0$ such that the sequences $\vec{m}$, $\vec{n}$, $\langle m_{i+1} - m_i \mid i \in \omega \rangle$ and $\langle n_{i+1} - n_i \rangle$ are strictly increasing. If $D_{\vec{m}} \leq D_{\vec{n}}$, then $(\vec{m}, \vec{n}) \in E_{\text{tail}}$.*

Proof. We choose x^*, r and X_0 as above. Suppose that $F\colon X \to X$ is a continuous map that reduces $D_{\vec{m}}$ to $D_{\vec{n}}$. Suppose that $F(x^*) \in D_i^{\vec{n}}$. Then there is an unfolding $\xi = \langle I, f, \bar{f} \rangle$ of $\langle H_{\vec{m}}, H_{\vec{n}} \rangle$with a base point for i, $V(I) = \{\langle 0, 1 \rangle\}$, $f(\langle 0, 1 \rangle) = \langle k, k+1 \rangle$ and $\bar{f}(\langle 0, 1 \rangle = \langle 0, 1 \rangle$. We consider the set M_i defined in Definition 2.12. The set M_i is a closed-open subset of X_0 by Lemma 2.14 and Lemma 2.15. It follows from this and from the choice of x^*, r and X_0 that for every $k \in \omega$, there is some $x^k \in M_i$ with $d(x^*, x) = r_k$ and the following properties.

(a) There is a sequence $\langle x_j \mid j \in \omega \rangle$ in X_0 with $\lim_j x_j = x^k$ and $d(x^*, x_j) < r_k$ for all $j \in \omega$.
(b) There is a sequence $\langle y_j \mid j \in \omega \rangle$ in X_0 with $\lim_j y_j = x^k$ and $d(x^*, y_j) > r_k$ for all $j \in \omega$.

It then follows from the Definition 2.12 of M_n and from Lemma 2.16 that there are $k_0 \in \omega$ and $l_i \in \mathbb{Z}$ such that for all $k \geq k_0$, we have $d(x^*, F(x^k)) = r_{k+l_i}$. We now choose some $j_0 \in \omega$ with $m_{j_0} \geq k_0$. Since F is a reduction of $D_{\vec{m}}$ to $D_{\vec{n}}$, since $d(x^*, x^k) = r_k$, $d(x^*, F(x^k)) = r_k + l_i$ and by the Definition 2.4 of the sets $D_j^{\vec{m}}$ and $D_j^{\vec{n}}$, there is some $m \in \mathbb{Z}$ such that for all $j > j_0$, $m_j = n_j + m$. Hence $(\vec{m}, \vec{n}) \in E_{\text{tail}}$. □

A *perfect subset* of ${}^\omega\omega$ is equal to the set of branches of a perfect tree, i.e. a subtree of ${}^{<\omega}\omega$ with cofinally many splitting nodes.

Proof of Theorem 1.2. We consider the equivalence relation E_{tail} restricted to the set A of sequences $\vec{n} = \langle n_i \mid i \in \omega \rangle$ such that $\vec{n}$ and $\langle n_{i+1} - n_i \mid i \in \omega \rangle$ are strictly increasing. Since the equivalence classes of $E_{\text{tail}} \upharpoonright A$ are countable and E_{tail} is Borel, there is a perfect subset B of A such that for all $\vec{m} \neq \vec{n}$ in B, $(\vec{m}, \vec{n}) \notin E_{\text{tail}}$. We define the Borel code of $D_{\vec{n}}$ as a sequence of open balls coding $D_{\vec{n}}$. The claim follows from Lemma 2.18. □

The conclusion of Theorem 1.2 is optimal in the sense that it is not possible to embed other configurations of Borel sets into the Wadge order of arbitrary Polish spaces of positive dimension.

Remark 2.19. There is a compact connected subset of X of $\mathbb{R}^3$ such that any two subsets of X that are non-trivial, i.e. nonempty and not equal to X, are incomparable in the Wadge order for X [Coo67, Theorem 11] (see [MRSS15, Remark after Theorem 5.15]).

Remark 2.20. There is an infinite-dimensional countable quasi-Polish space (X, d) such that the Wadge order on the Borel subsets of X is well-founded and satisfies the semilinear ordering principle (SLO) [MRSS15, Remark 5.35]. Hence Theorem 1.2 does not hold for all separable topological spaces.

Theorem 1.2 has the following application.

Example 2.21. There is a perfect set of Borel codes for distinct subsets of the complete Erdös space [DvM09] that are pairwise incomparable with respect to continuous reducibility. The complete Erdös space is totally disconnected.

Theorem 1.2 implies the characterization of Polish spaces of dimension 0 in Theorem 1.4 as follows.

Proof of Theorem 1.4. Suppose that (X, d) is Polish metric space of dimension 0. It is a standard result that (X, d) is homeomorphic to the set of branches $[T]$ of a subtree T of ${}^{<\omega}\omega$. It follows from

Borel determinacy that the Wadge order on $[T]$ satisfies Wadge's Lemma. Moreover the Wadge order on $[T]$ is well-founded (see [And07, Theorem 8]). This implies the remaining conditions.

All other implications follow from Theorem 1.2. $\square$

The following result provides a sufficient condition for embedding more configurations into the Wadge order on the Borel subsets of a metric space.

Lemma 2.22. *Suppose that $\langle X, d\rangle$ is a metric space. Suppose that $\langle X_n \mid n \in \omega\rangle$ is a partition of X into closed-open subsets such that each X_n has positive dimension. Let $P(\omega)$ denote the power set of ω. Then there is an order-preserving embedding of $\langle P(\omega), \subseteq\rangle$ into the Wadge order on the Borel subsets of X.*

Proof. For every $n \in \omega$ and every sequence $\vec{m} = \langle m_i \mid i \in \omega\rangle$ of natural numbers such that $\vec{m}$ and $\langle m_{i+1} - m_i \mid i \in \omega\rangle$ are strictly increasing, we construct a subset $D^n_{\vec{m}}$ of X_n as in the proof of Theorem 1.2. Suppose that $\langle \vec{m}_i \mid i \in \omega\rangle$ is a sequence of such sequences that are pairwise not E_{tail}-equivalent. For every subset I of ω, we define $D_I = \bigcup_{i\in I} D^i_{\vec{m}_i}$. Analogous to the proof of Theorem 1.2, $I \not\subseteq J$ implies that $D_I \not\leq D_J$. Moverover $D_I \leq D_J$ if $I \subseteq J$. $\square$

3. Incomparable sets of arbitrary complexity

The construction in the proof of Theorem 1.2 is used in the following proof.

Proof of Theorem 1.5. The assumption states that for every $x^* \in X$, there is some open set U containing x^* with compact closure. Since (X, d) has positive dimension, there is some $x^* \in X$ such that there is no neighborhood base at x^* that consists of closed-open sets. Suppose that U is an open ball containing x^* such that the closure of U is compact.

Let A denote the set of $x \in X$ such that there is a neighborhood base at x that consists of closed-open sets. Let $B = X \setminus A$.

Claim 3.1. *The set $\mathrm{cl}(B) \cap U$ is uncountable.*

Proof. Suppose that $\mathrm{cl}(B) \cap U$ is countable. The set $U \setminus \mathrm{cl}(B) \subseteq A$ has dimension 0 by the definition of A. Then U has dimension 0 by [HW41, Theorem II.2], contradicting the choice of U. $\square$

Since (X, d) is locally compact, it is complete. Hence there is a perfect subset C of $\mathrm{cl}(B) \cap U$ that is nowhere dense in $\mathrm{cl}(B) \cap U$ and a homeomorphism $f\colon {}^\omega 2 \to C$.

Claim 3.2. *There is a sequence $\langle x^*_n \mid n \in \omega\rangle$ of distinct elements of U, a sequence $\langle r_n \mid n \in \omega\rangle$ of positive real numbers and a set E with the following properties for all $m \neq n$ in ω.*

(a) $\mathrm{cl}(B_{r_n}(x^*_n)) \subseteq U$.
(b) $B_{r_m}(x^*_m) \cap B_{r_n}(x^*_n) = \emptyset$.
(c) $B_{r_m}(x^*_m) \cap C = \emptyset$.
(d) $x^*_n \in B$.
(e) $\lim_n r_n = 0$.
(f) $E = \{x^*_n \mid n \in \omega\}$.
(g) $C \subseteq \mathrm{cl}(E)$.

Proof. Since C is nowhere dense in $\mathrm{cl}(B) \cap U$, it is straightforward to construct the sequences by induction. $\square$

Suppose that $\langle \vec{m}_i \mid i \in \omega\rangle$ is a sequence of sequences that are pairwise not E_{tail}-equivalent with the following property. If $\vec{m}_i = \langle m^j_i \mid j \in \omega\rangle$, then the sequences $\vec{m}_i$ and $\langle m^{j+1}_i - m^j_i \mid j \in \omega\rangle$ are strictly increasing.

Suppose that $\langle r^j_n \mid j \in \omega\rangle$ is a strictly increasing sequence of real numbers with supremum r_n and $r^0_n = 0$. We construct subsets $D^n_i = D^n_{\vec{m}_i}$ of $B_{r_n}(x^*_n)$ analogous to the proof of Theorem 1.2. We define $D_I = f[I] \cup \bigcup_{n\in\omega} D^n_n$.

Suppose that $I \neq J$ are subsets of ${}^\omega 2$. We show that $A_I \not\leq A_J$. To this end, suppose that $F\colon X \to X$ is continuous and $D_I = F^{-1}[D_J]$.

Claim 3.3. *Suppose that $m \in \omega$. Then $F(x^*_m) \in D^n_n$ for some $n \in \omega$.*

Proof. Suppose that $F(x^*_m) \in C$. Let C_i denote the set of $x \in B_{r_m}(x^*_m)$ with $d(x^*_m, x) = r^i_m$. For every closed-open subset W of X containing $F(x^*_m)$, the set $F^{-1}[W]$ has an element x_i in C_i. Since C_i is closed and hence compact, this implies that for every $i \in \omega$, there is an element y_i of C_i with $F(y_i) = F(x^*_n)$. This contradicts the assumption that F is a reduction of D_I to D_J and the definition of the set D^m_m. □

Analogous to the proof of Theorem 1.2, the last claim implies that $m = n$. Since $C \subseteq \mathrm{cl}(E)$, this implies that $F \upharpoonright C = \mathrm{id} \upharpoonright C$. Since $D_I = F^{-1}[D_J]$, this implies $I = J$, contradicting the assumption. This completes the proof. □

4. Open questions

The set $D_{\vec{n}}$ defined in the proof of Theorem 1.2 are intersections of open and closed sets. This suggests the following question.

Question 4.1. *Does Theorem 1.2 hold for sets that are both an intersection of an open with a closed set and a union of an open set with a closed set?*

Question 4.2. *Does the proof of Theorem 1.2 work for separable normal spaces instead of metric spaces?*

Theorem 1.5 leaves open whether the assumption that the metric space (X, d) is locally compact can be dropped.

Question 4.3. *Does Theorem 1.5 hold for all Polish spaces of positive (small inductive) dimension?*

We further ask whether the results have analogues for functions on metric spaces instead of subsets of metric spaces (see [Car13, Ele02]).

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